Flat Sand Piles (work in progress)

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Abstract

Our Interest

A more realistic sand pile model?

Goal

characterization, properties, exhaustive generation ...
Flat Sand Piles are integer sequences which describe the states of a simple discrete dynamical system

Initial state: \((n), n\) sand grains in column 1,

Rule: in \((s_1, \ldots, s_l)\) a grain can fall from column \(i\) downto \(i + 1\) iff the height difference is at least 2 and maximal,

\[ s_i - s_{i+1} \geq 2 \land s_i - s_{i+1} \geq s_j - s_{j+1}, \quad 1 \leq i, j \leq l \]
Flat Sand Piles: moves

\[ \Delta s = (2, 3, 2, 1, 3, 2) \]

\[ s = (13, 11, 8, 6, 5, 2) \]
Notations and Definitions

- Potential at $i$: $\delta_i(s) = s_i - s_{i+1}$
- Potential vector: $\Delta(s) = (\delta_1(s), \ldots, \delta_l(s))$
- Potential of $s$: $P(s) = \max(\Delta(s))$
- $s <_{nlex} t$ iff $\exists i: s_j = t_j, j < i, s_i > t_i$
- The triangular length of $n$ is $t(n) = p$ s.t.
  \[
  \frac{(p - 1)p}{2} < n \leq \frac{(p + 1)p}{2}, \quad t(n) = O(\sqrt{n})
  \]
- The triangular bit representation of $n$ is $T(n) = (b_1, \ldots, b_p)$ s.t. $n = \sum_{i=1}^{p} b_i i$, $b_i \in \{0, 1\}$ and
  - $p = t(n)$, $b_p = 1$;
  - $b_i = 0$ for $q < i < p$ where $q = t(n - p)$;
  - $(b_1, \ldots, b_q) = T(n - p)$. 

Triangular bit representation

$T(n)$ can be obtained as follows:

1. $p = t(n)$
2. $k = n - \frac{(p-1)p}{2}$
3. 

$$T(n) = \begin{cases} 
1^p & \text{if } k = p \\
1^{p-k-1}01^k & \text{otherwise}
\end{cases}$$

Example

For $n = 17$ one has $p = 6$, $k = 2$. Then,

$$T(17) = 1^301^2 = (1, 1, 1, 0, 1, 1) \approx 1 + 2 + 3 + 5 + 6 = 17$$
Flat Sand Piles: moves

**Definition (move)**

Let $p = \Delta(s)$, $D = P(s)$. Then $\text{Move}(s, i) = t$, $s \xrightarrow{i} t$, iff $p_i = D \geq 2$ and either

$$p = (p_1, \ldots, p_{i-1}, D, p_{i+1}, \ldots), \quad p_{i-1}, p_{i+1} < D$$

$$\Delta(t) = (p_1, \ldots, p_{i-1} + 1, D - 2, p_{i+1} + 1, \ldots)$$

or

$$p = (p_1, \ldots, p_{i-1}, D, \ldots, D, p_{i+k}, \ldots), \quad p_{i-1}, p_{i+k} < D, k > 1$$

$$\Delta(t) = (p_1, \ldots, p_{i-1} + 1, D - 1, D, \ldots, D, D - 1, p_{i+1} + 1, \ldots)$$
FSPM($n$): a move

Move(s,2)
FSPM($n$) is the closure of \{(n)\} under Move

- A (right?) model to study high likely states of physical phenomena (e.g. avalanches)
- Is a lattice
- FSPM($n$) $\subset$ SPM($n$)
For $n = 10$ one has
FSPM(\(n\)): a bigger example
Theorem

Let \( s \in SPM(n) \), \( D = P(s) \) and \( p = \Delta(s) \). Then \( s \in FSPM(n) \) iff

1. \( \forall i, 1 \leq i < l(s), \ D - p_i \leq 2; \)
2. \( \forall k \geq 0, \ p \neq x \cdot (D - 2) \cdot (D - 1)^{[k]} \cdot (D - 2) \cdot y \) with \( y \neq \epsilon; \)
FSPM\( (n) \): some results

Fact

- If \( s \in FSPM_r(n) \) then \( E \leq P(s) \leq D \) where
  \[
  D = \min\{j|n \leq (j - 1) \frac{r(r - 1)}{2} + 2\}
  \]
  \[
  E = \max\{j|j \frac{r(r + 1)}{2} \leq n\}
  \]

- If \( s, t \in FSPM_r(n) \) and \( P(s) > P(t) \) then \( s <_{nlex} t \)

Corollary

\( \min <_{nlex}(FSPM_r(n)) \) has the highest potential in \( FSPM_r(n) \)
Lemma

Let \( \alpha = \frac{(r-1)(r-2)}{2} \), \( \beta = \frac{r(r-1)}{2} \). Then \( s = \min_{\text{nlex}}(\text{FSPM}_r(n)) \) is defined by

\[
\Delta(s) = \begin{cases} 
D \cdot (D - 1)^{[r-3]} \cdot (D - 2, 1) & k = 0 \\
f(T(k + 1)) \cdot (D - 1)^[b] \cdot (D - 2, 1) & 1 \leq k < \alpha \\
D^{[r-3]} \cdot (D - 1, D - 1, 1) & k = \alpha \\
D^{[r-a-3]} \cdot (D - 1) \cdot D^[a] \cdot (D - 1, 1) & \alpha < k < \beta - 1 \\
D^{[r-2]} \cdot (D - 1, 1) & k = \beta - 1 
\end{cases}
\]

where \( (D - 1)^\beta + 2 \leq n < D\beta + 2 \), \( k = n - (D - 1)^\beta - 2 \), \( b = r - 2 - t(k + 1) \), \( a = k - \alpha \) and \( f : \{0, 1\}^* \mapsto \{D, D - 1\}^* \) is the mapping defined by \( f(0) = D - 1 \), \( f(1) = D \)
Example

Let $s = \min_{<_{\text{nlex}}} (\text{FSPM}_r(n))$ and $r = 4$. Then $\alpha = 3, \beta = 6$.

- for $n = 20$ one has $D = 4, k = 0$ and

  $s = (10, 6, 3, 1), \quad \Delta(s) = (4, 3, 2, 1)$

- if $n = 40$ one has $D = 7, k = 2, f(T(3)) = f(1, 1) = (7, 7), b = 0$ and

  $s = (20, 13, 6, 1), \quad \Delta(s) = (7, 7, 5, 1)$

- for $n = 41$ one has $D = 7, k = 3$ and

  $s = (20, 13, 7, 1), \quad \Delta(s) = (7, 6, 6, 1)$
Lemma

Let \( A = \{ v \in FSPM_r(n) | v_r = P(v) \} \). Then

- if \( s, t \in A \) then \( P(s) = P(t) \)
- if \( s, t \in A \) and \( s = \min <_{nlex}(A) \) then \( s \xrightarrow{\star} t \)

Corollary

Let \( s = \min <_{nlex}(FSPM_r(n)) \). Then,

\[
FSPM_r(n) = G(s) = \{ t \in FSPM_r(n) | s \xrightarrow{\star} t \}
\]
Lemma

Let \( A = \{ v \in FSPM_r(n) | v_r = P(v) \} \). Then

- if \( s, t \in A \) then \( P(s) = P(t) \)
- if \( s, t \in A \) and \( s = \min_{\text{nlex}}(A) \) then \( s \rightarrow^* t \)

Corollary

Let \( s = \min_{\text{nlex}}(FSPM_r(n)) \). Then,

\[
FSPM_r(n) = G(s) = \{ t \in FSPM_r(n) | s \rightarrow^* t \}
\]
Does the approach used to generate IPM$_k(n)$ work also for FSPM($n$)?

**Advantage:** we can directly generate FSPM$_r(n)$ starting from $\min <_{n\text{lex}}(\text{FSPM}_r(n))$. Let $M(s) = \{j|\text{Move}(s,j) \neq \bot\}$.

1. Define the grand ancestor:

   \[ A(s) = \min <_{n\text{lex}} \{ t \in \text{FSPM}(n) | t_{\leq i} = s_{\leq i}, i \in M(t), i = \max(M(s)) \} \]

2. Prove that for all $e > 0$

   \[ A(s^{(e)}) \xrightarrow{i_e} s^{(e+1)}, \quad i_e = \max(M(s^{(e)})) \]
Conclusions

FSPM($n$) is just a starting point to investigate other more interesting (e.g. probabilistic) models extending SPM($n$)

What about 2D-FSPM($n$)?

Simulation: http://www.dsi.unifi.it/users/brocchi/dev/ms.html
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